Comparison of Numerical Methods in Uncertainty Quantification

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Outline

- **Introduction**

- Why *Uncertainty Quantification*(UQ)?;
- Motivation;
- UQ and *Computational Science*.

- **Deterministic model**

-second order oscillator

$$\begin{cases} x'(t) = y(t) \\ y'(t) + c \cdot y(t) + k \cdot x(t) = F \cdot cos(\omega_a \cdot t) \\ x(0) = y(0) = 0, \end{cases} \quad (1)$$

where $\omega_a = 1.05 \; rad/s$, $c = 5 \; [N \cdot s/m]$ and $F = 0.1 \; [N/m^2]$.

. . . with analitical solution:

$$x(t,k) = c_1(k) \cdot \cos(\omega_a \cdot t) + c_2(k) \cdot \sin(\omega_a \cdot t) + c_3(k) \cdot e^{\lambda_1(k) \cdot t} + c_4(k) \cdot e^{\lambda_2(k) \cdot t} \quad (2)$$

- **Stochastic model**

-in (1) $\rightarrow$ uncertainty in k $\rightarrow$ Gaussian random variable $K \sim N(0,1)$, i.e. its PDF is:

$$\rho : \mathbb{R} \rightarrow \mathbb{R}^{+}, \rho(x) = \frac{1}{\sqrt{2\pi}} e^{\frac{-x^2}{2}} \tag{3}$$

- **Monte Carlo sampling**

- sampling based method;
- convergence $\sim \mathcal{O}(\frac{1}{\sqrt{n}})$;
- non-intrusive.

1. Generate n iid samples of the random inputs, where n is user defined;

2. For each generated sample, solve the associated stochastic equation and obtain the solution $x^{(i)}(t, \omega)$;

3. Estimate the required solution statistics, e.g. expectation(mean) as $\mathbb{E}[x^{(i)}(t, \omega)] = \frac{1}{n} \sum_{i=1}^{n} x^{(i)}(t, \omega^{(i)})$ and variance as $Var[x^{(i)}(t, \omega)] = \frac{1}{n-1} \sum_{i=1}^{n} (x^{(i)}(t, \omega^{(i)}) - \mathbb{E}[x^{(i)}(t, \omega)])^2$

Generalized Polynomial Chaos(gPC) methods

- approximate the solution via orthogonal polynomials;
- PDF $\rightarrow$ polynomials;
- Gaussian random variables $\rightarrow$ Hermite polynomials.

$$x(t,\omega) \approx x^Q(t,\omega) = \sum_{j=0}^{Q} c^j(t) \cdot \phi^j(\omega), \tag{4}$$

$$c^j(t) = \int_{\mathbb{R}} x(t,\omega) \cdot \phi^j(\omega) \cdot \rho(\omega) \cdot d\omega = \mathbb{E}[x(t,\omega) \cdot \phi^j(\omega)], j = 0 \ldots Q \tag{5}$$

- **Stochastic Collocations**

- set of nodes(collocation points);
- coefficiants $c^j(t)$ of the gPC expansion $\rightarrow$ quadrature rule;
- Gaussian random variable $\rightarrow$ Gauss-Hermite quadrature;
- exponential convergence in Q(the order of the gPC expansion);
- non-intrusive.

$$c^j(t) = \int_{\mathbb{R}} x(t,\omega) \cdot \phi^j(\omega) \cdot \rho(\omega) d\omega \approx \sum_{i=0}^{N} x(t,\zeta^i) \cdot \phi^j(\zeta^i) \cdot \alpha^i \quad (6)$$

- **Stochastic Galerkin**

- stochastic `weak form`;
- `projection`;
- `coupled` system of ODEs;
- `exponential convergence` in Q(the order of the gPC expansion);
- `intrusive`.

- **Statistics**

$$\mathbb{E}[x(t,\omega)] = c^0 \quad (7)$$

$$Var[x(t,\omega)] = \sum_{j=1}^{Q} \left(\frac{c^j(t)}{h_j}\right)^2, \quad (8)$$

$$h_j = \int_{\mathbb{R}} \left(\phi^j(x)\right)^2 \cdot \rho(x) \cdot dx \quad (9)$$

- **Numerical error**

Monte Carlo sampling error

$$\epsilon_{MC} = \epsilon_{sampling} + \epsilon_{num\_solution} \quad (10)$$

Stochastic Collocations error

$$\epsilon_{SC} = \epsilon_{gPC\_approx} + \epsilon_{num\_quadrature} + \epsilon_{num\_solution} \quad (11)$$

Stochastic Galerkin error

$$\epsilon_{SG} = \epsilon_{gPC\_approx} + \epsilon_{num\_solution} \quad (12)$$

- **Numerical results - Setup**

- $t \in [0, 10]$;
- Quantity of Interest(QoI) - solution at $t_{interest}$ = 10s;
- statistical moments: expectation(mean) and variance;
- via the analytical solution:

$$\mathbb{E}[x(t_{interest}, \omega)] = 0.01023, \;\; Var[x(t_{interest}, \omega)] = 0.02128 \tag{13}$$

- **Absolute errors via numerical solution**

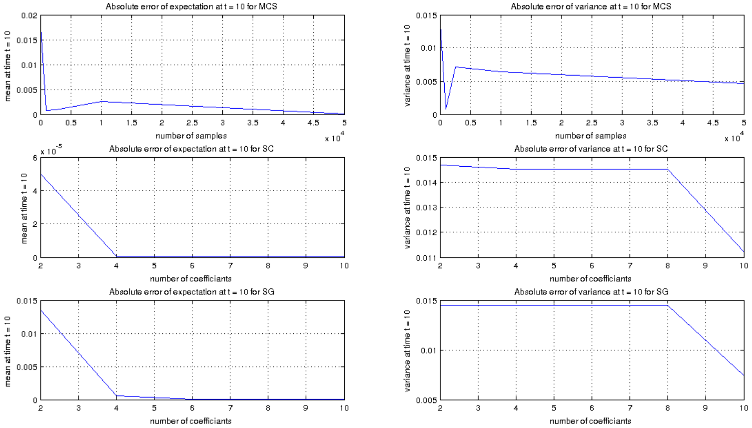

Figure: Results obtained via the numerical solution

- **Absolute errors via analytical solution**

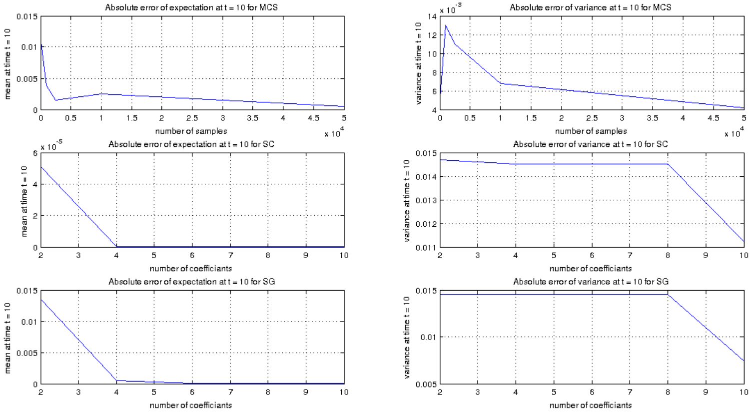

Figure: Results obtained via the analytical solution

- **Runtime - numerical&analytical solution**

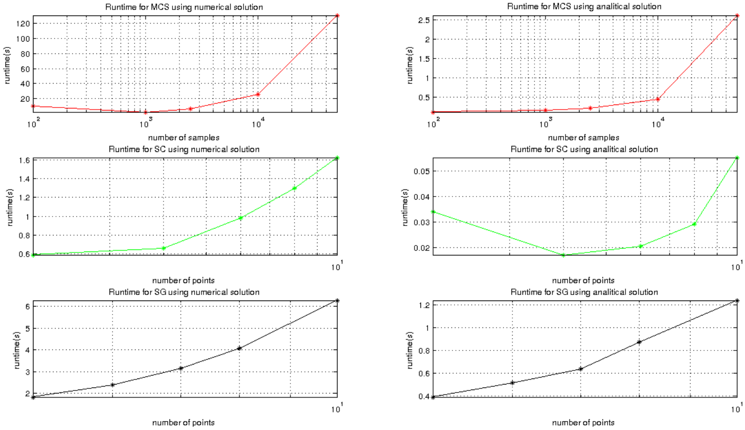

Figure: Runtime via the numerical and analytical solution

- **Conclusions**

- intrusive vs. non-intrusive methods;
- 1d stochastic system;
- comparison with the analytical solution;
- superiority of the gPC methods;
- further developement.

Thank you for your attention!

Any questions?